Non-deterministic Semantics as a Proof-Theoretical Tool

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Vienna University of Technology

Joint work with Matthias Baaz and Ori Lahav
Our goals:

- Characterization of important syntactic properties of calculi: *cut-admissibility, the subformula property, invertibility of rules,...*
- Understanding the dependencies between them.

Our tool: non-deterministic semantics.

Our case study: canonical labelled calculi.
Can we semantically characterize $\vdash G - (cut) s$? For example, what is the semantics of the logic induced by $\text{LK} - (cut)$?
Can we semantically characterize $\vdash_{G-\text{(cut)}} S$?
Can we semantically characterize $\vdash_{G-\text{(cut)}}$?

For example, what is the semantics of the logic induced by $\text{LK} - (\text{cut})$?
What is a logic?

1. A formal language \( \mathcal{L} \), based on which \( \mathcal{L} \)-formulas are constructed.

2. A relation \( \vdash \) between sets of \( \mathcal{L} \)-formulas and \( \mathcal{L} \)-formulas, satisfying:

   - Reflexivity: if \( \psi \in \mathcal{T} \) then \( \mathcal{T} \vdash \psi \).
   - Monotonicity: if \( \mathcal{T} \vdash \psi \) and \( \mathcal{T} \subseteq \mathcal{T}' \), then \( \mathcal{T}' \vdash \psi \).
   - Transitivity: if \( \mathcal{T} \vdash \psi \) and \( \mathcal{T}', \psi \vdash \varphi \) then \( \mathcal{T}, \mathcal{T}' \vdash \varphi \).
How are logics defined by sequent calculi?

Sequent calculi can induce logics in two possible ways:

- **v:** $\Gamma \vdash^v_G \varphi \iff \{ \Rightarrow \psi \mid \psi \in \mathcal{T} \} \vdash^G \Rightarrow \varphi$
- **t:** $\Gamma \vdash^t_G \varphi \iff \vdash^G \Gamma \Rightarrow \varphi \text{ for some finite } \Gamma \subseteq \mathcal{T}$

*Lemma*

For any sequent calculus $G$, $\vdash^v_G$ is a logic. But if $G$ does not include cut, $\vdash^t_G$ is not necessarily a logic!
How are logics defined by sequent calculi?

- Sequent calculi can induce logics in two possible ways:
  
  \[ \begin{align*}
  v: & \quad \mathcal{T} \vdash^v_G \varphi \iff \{ \Rightarrow \psi \mid \psi \in \mathcal{T} \} \vdash_G \Rightarrow \varphi \\
  t: & \quad \mathcal{T} \vdash^t_G \varphi \iff \vdash_G \Gamma \Rightarrow \varphi \quad \text{for some finite } \Gamma \subseteq \mathcal{T}
  \end{align*} \]

**Lemma**

For any sequent calculus \( G \), \( \vdash^v_G \) is a logic.

But if \( G \) does not include cut, \( \vdash^t_G \) is not necessarily a logic!
Cut-Admissibility

Can we semantically characterize the logic $\vdash_{v \text{ LK}} (\text{cut})$?

- $\vdash_{v \text{ LK}}$ and $\vdash_{v \text{ LK}-(cut)}$ are different logics:

  $\Rightarrow p_1 \supset p_2 \vdash_{\text{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$

  $\Rightarrow p_1 \supset p_2 \vdash_{v \text{ LK}-(cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$
Can we semantically characterize the logic $\vdash_{\mathbf{LK}-(cut)}$?

- $\vdash_{\mathbf{LK}}$ and $\vdash_{\mathbf{LK}-(cut)}$ are different logics:

  $$\Rightarrow p_1 \supset p_2 \vdash_{\mathbf{LK}} \Rightarrow p_1 \supset (p_3 \supset p_2)$$

  $$\Rightarrow p_1 \supset p_2 \not\vdash_{\mathbf{LK}-(cut)} \Rightarrow p_1 \supset (p_3 \supset p_2)$$
Classical Logic

The Matrix \( \mathbf{MLK} \)

- Truth-values: \( \{T, F\} \)
- An \( \mathbf{MLK} \)-valuation is a *model* of a sequent \( \Gamma \Rightarrow \Delta \) iff \( v(\psi) = F \) for some \( \psi \in \Gamma \) or \( v(\psi) = T \) for some \( \psi \in \Delta \).
- Truth-tables:

<table>
<thead>
<tr>
<th>( \neg )</th>
<th>T</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>F</td>
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<td>T</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Soundness and Completeness

\( \Omega \vdash_{LK} s \) iff every \( \mathbf{MLK} \)-valuation which is a model of every sequent in \( \Omega \) is also a model of \( s \).
Classical Logic

The Matrix $\mathbf{MLK}$

- Truth-values: $\{T, F\}$
- An $\mathbf{MLK}$-valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $v(\psi) = F$ for some $\psi \in \Gamma$ or $v(\psi) = T$ for some $\psi \in \Delta$.
- Truth-tables:

\[
\begin{array}{c|c|c}
\sim & T & F \\
\hline
T & T & F \\
F & T & T \\
\end{array}
\quad
\begin{array}{c|c|c}
\land & T & F \\
\hline
T & T & F \\
F & F & F \\
\end{array}
\]

Soundness and Completeness

$\Omega \vdash_{\mathbf{LK}} s$ iff every $\mathbf{MLK}$-valuation which is a model of every sequent in $\Omega$ is also a model of $s$.

(Trivial) Observation

Every $\mathbf{MLK}$-valuation $v$ is either a model of $\Rightarrow \varphi$ or of $\varphi \Rightarrow$, but not both!
The semantics for \( \Vdash_{\text{LK}}^\nu \) (cut)

(Trivial) Observation

Every \( M_{\text{LK}} \)-valuation \( \nu \) is either a model of \( \Rightarrow \varphi \) or of \( \varphi \Rightarrow \), but not both!

- Why not both? Because of cut:
  \[
  \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}
  \]

- Discarding cut makes this option possible.
The semantics for $\vdash^v_{LK-(cut)}$

(Trivial) Observation
Every $M_{LK}$-valuation $v$ is either a model of $\Rightarrow \varphi$ or of $\varphi \Rightarrow$, but not both!

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  \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}
  \]
- Discarding cut makes this option possible.
- New truth-values: $\{\{T\}, \{F\}, \{T, F\}\}$
The semantics for \( \vdash^\nu_{LK-(cut)} \)

### (Trivial) Observation

Every \( M_{LK} \)-valuation \( \nu \) is either a model of \( \Rightarrow \varphi \) or of \( \varphi \Rightarrow \), but not both!

- Why not both? Because of cut: 
  \[
  \Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta \\
  \hline
  \Gamma \Rightarrow \Delta
  \]
- Discarding cut makes this option possible.
- **New truth-values:** \{\{T\}, \{F\}, \{T, F\}\}
- **New definition of model:** A valuation is a *model* of a sequent \( \Gamma \Rightarrow \Delta \) iff 
  \( F \in \nu(\psi) \) for some \( \psi \in \Gamma \) or \( T \in \nu(\psi) \) for some \( \psi \in \Delta \).
  - For example: \( \nu(\varphi) = \{T, F\} \) iff \( \nu \) is a model of both \( \Rightarrow \varphi \) and \( \varphi \Rightarrow \).
The semantics for $\vdash^v_{\text{LK}-(\text{cut})}$

(Trivial) Observation

Every $M_{\text{LK}}$-valuation $v$ is either a model of $\Rightarrow \varphi$ or of $\varphi \Rightarrow$, but not both!

- Why not both? Because of cut: $
\frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

- Discarding cut makes this option possible.

- New truth-values: $\{\{\text{T}\},\{\text{F}\},\{\text{T},\text{F}\}\}$

- New definition of model: a valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $F \in v(\psi)$ for some $\psi \in \Gamma$ or $T \in v(\psi)$ for some $\psi \in \Delta$.
  - For example: $v(\varphi) = \{\text{T},\text{F}\}$ iff $v$ is a model of both $\Rightarrow \varphi$ and $\varphi \Rightarrow$.

- But no new truth-tables!

Theorem

(Lahav, 2012) $\vdash^v_{\text{LK}-(\text{cut})}$ does not have a finite characteristic matrix.
Our goals:
- Characterization of important syntactic properties of calculi.
- Understanding the dependencies between them.

Our tool: non-deterministic semantics.

Our case study: canonical labelled calculi.
Non-deterministic Semantics - Motivation

- **Principle of Truth-Functionality (PTF):** the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas.

- **Non-deterministic phenomena in possible conflict with PTF:**
  - vagueness
  - incompleteness
  - uncertainty
  - imprecision
  - inconsisteny

- **Relaxing PTF:** non-deterministic evaluation of formulas.

<table>
<thead>
<tr>
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<th>T</th>
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<tbody>
<tr>
<td>T</td>
<td>{T}</td>
<td>{T, F}</td>
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<tr>
<td>F</td>
<td>{T, F}</td>
<td>{F}</td>
</tr>
</tbody>
</table>
Intuition for Introducing Non-determinism

Consider a fully structural calculus with the following rules:

\[
\begin{align*}
&\Gamma \Rightarrow \Delta, \psi \\
&\Gamma, \neg\psi \Rightarrow \Delta \\
&\Gamma, \psi \Rightarrow \Delta \\
&\Gamma, \varphi \Rightarrow \Delta \\
&\Gamma, \psi \lor \varphi \Rightarrow \Delta \\
&\Gamma \Rightarrow \Delta, \psi, \varphi \\
&\Gamma \Rightarrow \Delta, \psi \lor \varphi
\end{align*}
\]
Intuition for Introducing Non-determinism

\[ \Gamma \Rightarrow \Delta, \psi \]
\[ \Gamma, \neg \psi \Rightarrow \Delta \]
\[ \Gamma, \psi \Rightarrow \Delta \]
\[ \Gamma, \varphi \Rightarrow \Delta \]
\[ \Gamma, \psi \lor \varphi \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta, \psi \lor \varphi \]

\[ \neg \]
\[ T \quad F \]
\[ T \quad F \]
\[ F \quad T \]

\[ \lor \]
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\[ T \quad F \quad T \]
\[ F \quad T \quad T \]
\[ F \quad F \quad F \]
Intuition for Introducing Non-determinism

\[
\begin{align*}
\Gamma \Rightarrow \Delta, \psi \\
\Gamma, \neg \psi \Rightarrow \Delta \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow \Delta, \psi, \phi \\
\Gamma \Rightarrow \Delta, \psi \lor \phi \\
\end{align*}
\]

\[
\begin{array}{c|cc|}
\top & \top & \top \\
\top & \top & \top \\
\top & \top & \top \\
\top & \top & \top \\
\end{array}
\]
Intuition for Introducing Non-determinism

\[
\begin{align*}
\Gamma \Rightarrow \Delta, \psi & \\
\Gamma, \neg \psi \Rightarrow \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow \Delta, \psi, \varphi & \\
\Gamma \Rightarrow \Delta, \psi \vee \varphi
\end{align*}
\]
Many-valued Matrices

A (deterministic) matrix $\mathbf{M}$ for $\mathcal{L}$ consists of:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{V}$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$.

An $\mathbf{M}$-valuation $\nu : \text{Frm}_\mathcal{L} \rightarrow \mathcal{V}$ satisfies:

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$$
A non-deterministic matrix $\mathbf{M}$ for $\mathcal{L}$ consists of:

- $\mathcal{V}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow \mathcal{P}^+(\mathcal{V})$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$.

An $\mathbf{M}$-valuation $\nu : \text{Frm}_\mathcal{L} \rightarrow \mathcal{V}$ satisfies:

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$$
Example: The Paraconsistent Logic CLuN of Batens

$L$ — a language over \{\lor, \land, \supset, \neg\}, $V = \{F, T\}, $D = \{T\}.

\lor, \land$ and $\supset$ are interpreted classically, while $\neg$ satisfies the law of excluded middle $\neg\varphi \lor \varphi$, but not the law of contradiction $\neg(\varphi \land \neg\varphi)$.

$M^2 = \langle V, D, O \rangle$ where $O$ is given by:

\[
\begin{array}{c|c|c|c}
\lor & \land & \supset \\
\hline
T & T & \{T\} & \{T\} & \{T\} \\
T & F & \{T\} & \{F\} & \{F\} \\
F & T & \{T\} & \{F\} & \{T\} \\
F & F & \{F\} & \{F\} & \{T\} \\
\end{array}
\]

\[
\begin{array}{c|c}
\neg & \\
\hline
T & \{T, F\} \\
F & \{T\} \\
\end{array}
\]
Key property of Nmatrices:

- **Analyticity**: any partial $\mathbf{M}$-valuation can be extended to a full $\mathbf{M}$-valuation.
- **Consequence**: decidability (in the finite case).
What is the semantics of $\vdash^\gamma_{LK-(cut)}$?

- We start with the simplest system: identity axiom + weakening (no logical rules, no cut)
- Truth-values: $\{\{T\}, \{F\}, \{T, F\}\}$
What is the semantics of $\vdash_{\text{LK}}^\forall (\text{cut})$?

- We start with the simplest system: identity axiom + weakening (no logical rules, no cut)
- Truth-values: $\{\{T\}, \{F\}, \{T, F\}\}$

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\wedge$</th>
<th>${T}$</th>
<th>${F}$</th>
<th>${T, F}$</th>
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<tbody>
<tr>
<td>${T}$</td>
<td>${{T}, {F}, {T, F}}$</td>
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</table>
What is the semantics of $\vdash_{LK}^{(cut)}$?

Adding the rule:

$$(\Rightarrow \land) \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \land \varphi}$$

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\land$</th>
<th>${T}$</th>
<th>${F}$</th>
<th>${T, F}$</th>
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</table>
What is the semantics of $\vdash^\lor_{LK\neg (cut)}$?

Adding the rule:

$$ (\Rightarrow \land) \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \land \varphi} $$

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\land$</th>
<th>${T}$</th>
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<td>${{T}, {T,F}}$</td>
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</table>
What is the semantics of $\vdash^{\gamma}_{LK} - (cut)$?

Adding the rule:

\[
(\wedge \Rightarrow) \quad \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}
\]

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\tilde{\wedge}$</th>
<th>${T}$</th>
<th>${F}$</th>
<th>${T, F}$</th>
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<tr>
<td>${T}$</td>
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<td>${{T}, {T, F}}$</td>
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</table>
What is the semantics of $\vdash_{\mathbf{LK}}\neg (\text{cut})$?

Adding the rule:

$$(\land \Rightarrow) \quad \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \land \varphi \Rightarrow \Delta}$$

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\land$</th>
<th>${\text{T}}$</th>
<th>${\text{F}}$</th>
<th>${\text{T, F}}$</th>
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<tbody>
<tr>
<td>${\text{T}}$</td>
<td>${{\text{T}}, {\text{T, F}}}$</td>
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<td>${{\text{T, F}}}$</td>
<td>${{\text{F}, {\text{T, F}}}$</td>
<td>${{\text{T, F}}}$</td>
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</table>
What is the semantics of \( \vdash_{LK \neg \text{(cut)}} \)?

The corresponding \( \tilde{\wedge} \) matrix:

<table>
<thead>
<tr>
<th>( \tilde{\wedge} )</th>
<th>{T}</th>
<th>{F}</th>
<th>{T, F}</th>
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<tbody>
<tr>
<td>{T}</td>
<td>{{T}, {T, F}}</td>
<td>{{F}, {T, F}}</td>
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<tr>
<td>{T, F}</td>
<td>{{T, F}}</td>
<td>{{F}, {T, F}}</td>
<td>{{T, F}}</td>
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</tbody>
</table>

Recall: An valuation is a **model** of a sequent \( \Gamma \Rightarrow \Delta \) iff \( f \in v(\psi) \) for some \( \psi \in \Gamma \) or \( t \in v(\psi) \) for some \( \psi \in \Delta \).
What is the semantics of $\vdash_{LK-(cut)}^v$?

The corresponding Nmatrix:

<table>
<thead>
<tr>
<th>$\wedge$</th>
<th>{T}</th>
<th>{F}</th>
<th>{T, F}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{T}</td>
<td>{{T}, {T, F}}</td>
<td>{{F}, {T, F}}</td>
<td>{{T, F}}</td>
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<tr>
<td>{F}</td>
<td>{{F}, {T, F}}</td>
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<td>{{F}, {T, F}}</td>
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<tr>
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<td>{{F}, {T, F}}</td>
<td>{{T, F}}</td>
</tr>
</tbody>
</table>

Recall: An valuation is a model of a sequent $\Gamma \Rightarrow \Delta$ iff $f \in v(\psi)$ for some $\psi \in \Gamma$ or $t \in v(\psi)$ for some $\psi \in \Delta$.

Soundness and Completeness

$\Omega \vdash_{LK-(cut)}^v s$ iff every $M_{LK-(cut)}$-valuation which is a model of every sequent in $\Omega$ is also a model of $s$.

→ New formulation of results of Schütte (1960) and Girard (1987).
Application: Semantic Proof of Cut-Admissibility in $\textbf{LK}$

Cut-Admissibility in $\textbf{LK}$

$\vdash_{\textbf{LK}} S \iff \vdash_{\textbf{LK}-(\text{cut})} S$

Reduces to proving that for every $M_{\textbf{LK}-(\text{cut})}$-valuation which is not a model of some sequent $s$, there exists an $M_{\textbf{LK}}$-valuation which is not a model of $s$.

Proof by induction on the build-up of formulas.
Reduces to proving that for every $M_{\text{LK}-(\text{cut})}$-valuation which is not a model of some sequent $s$, there exists an $M_{\text{LK}}$-valuation which is not a model of $s$.

Proof by induction on the build-up of formulas.
Our goals:

- Characterization of important syntactic properties of calculi.
- Understanding the dependencies between them.

Our tool: non-deterministic semantics.

Our case study: canonical labelled calculi.
What is a Canonical Rule?

- An “ideal” logical rule: an introduction rule for exactly one connective, on exactly one side of a sequent.
- In its formulation: exactly one occurrence of the introduced connective, no other occurrences of other connectives.
- Its active formulas: immediate subformulas of its principal formula.
Examples of Canonical Rules

\[\Gamma, \psi, \varphi \Rightarrow \Delta\]
\[\Gamma, \psi \land \varphi \Rightarrow \Delta\]
\[\Gamma \Rightarrow \Delta, \psi\]
\[\Gamma, \neg \psi \Rightarrow \Delta\]

\[\Gamma \Rightarrow \Delta, \psi\]
\[\Gamma \Rightarrow \Delta, \neg \psi\]
Example 1

Let $G_1$ be a fully structural calculus with the following rules:

$$\{\rightarrow \psi_1 ; \rightarrow \psi_2 \} / \psi_1 \diamond \psi_2 \Rightarrow \{\psi_1 \Rightarrow ; \psi_2 \Rightarrow \} / \Rightarrow \psi_1 \diamond \psi_2$$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$\diamond (a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>${F}$</td>
</tr>
<tr>
<td>T</td>
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<td>${T,F}$</td>
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<td>F</td>
<td>T</td>
<td>${T,F}$</td>
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<tr>
<td>F</td>
<td>F</td>
<td>${T}$</td>
</tr>
</tbody>
</table>
Let $G_2$ be a fully structural calculus with the following rules:

$$\{\psi_2 \Rightarrow\} / \psi_1 \circ \psi_2 \Rightarrow \{\Rightarrow \psi_1\} / \Rightarrow \psi_1 \circ \psi_2$$

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$(a, b)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>F</td>
<td>T</td>
<td>${T,F}$</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>${F}$</td>
</tr>
</tbody>
</table>
A non-deterministic matrix for $\mathcal{L}$ consists of:

- $\mathcal{T}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V})$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$. 
A non-deterministic partial matrix for $\mathcal{L}$ consists of:

- $\mathcal{T}$ - the set of truth-values,
- $\mathcal{O}$ - contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \rightarrow P(\mathcal{V})$ for every $n$-ary connective $\diamond$ of $\mathcal{L}$.

A PNmatrix is proper if it includes no “empty spots”.

Non-deterministic Partial Matrices
Key property of Nmatrices:

- **Analyticity**: any partial $M$-valuation can be extended to a full $M$-valuation.
- **Consequence**: decidability (in the finite case).
Key property of PNmatrices:

- **Weak Analyticity:** it is *decidable* whether a partial $M$-valuation can be extended to a full $M$-valuation.
- **Consequence:** decidability (in the finite case).
The two-sided case: a direct correspondence

Theorem

If \( G \) is a (two-sided) canonical calculus, then the following statements are equivalent:

1. \( G \) has a characteristic proper two-valued PNmatrix.
2. \( G \) enjoys strong cut-admissibility.
3. \( G \) enjoys the subformula property.
The two-sided case: a direct correspondence

**Theorem**

*If* $G$ *is a (two-sided) canonical calculus, then the following statements are equivalent:*

1. $G$ *has a characteristic proper two-valued PNmatrix.*
2. $G$ *enjoys strong cut-admissibility.*
3. $G$ *enjoys the subformula property.*

- **The Subformula Property:** Whenever $\Omega \vdash_G s$, there is a derivation of $s$ from $\Omega$ in $G$ consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).

- **Strong Cut-Admissibility** Whenever $\Omega \vdash_G s$, there is a derivation of $s$ from $\Omega$ in $G$ in which cuts are allowed only on formulas from $\Omega$. 
A finite set of labels $\mathcal{L}$.

A labelled formula: $a : \psi$ for $a \in \mathcal{L}$

A sequent: a finite set of labelled formulas.

Canonical labelled calculi have in addition to weakening two types of rules: primitive rules and canonical introduction rules.
Primitive Rules

\[(L_1 : \psi) \cup s \quad \ldots \quad (L_n : \psi) \cup s\]
\[\overline{(L : \psi) \cup s \cup \ldots \cup s}\]

Notation: we write \((\{a, b, c\} : \psi)\) instead of \(\{a : \psi, b : \psi, c : \psi\}\).

Examples:

\[\{F : \psi\} \cup s \quad \{T : \psi\} \cup s\]
\[\overline{s}\]

\[s\]
\[\overline{(\{T, F\} : \psi) \cup s}\]

\[\{(a) : \psi\} \cup s \quad \{(b) : \psi\} \cup s\]
\[\overline{(\{c, d\} : \psi) \cup s}\]
Canonical Introduction Rules

\[
\frac{\{T : \psi_1\} \cup s \quad \{T : \psi_2\} \cup s}{\{T : \psi_1 \land \psi_2\} \cup s}
\]

\[
\frac{\{F : \psi_1, F : \psi_2\} \cup s}{\{F : \psi_1 \land \psi_2\} \cup s}
\]

\[
\frac{\{a : \psi_1, b : \psi_2\} \cup s \quad \{c : \psi_2, a : \psi_3, b : \psi_3\} \cup s}{(\{a, b\} : \circ(\psi_1, \psi_2, \psi_3) \cup s)
\]
Possible truth-values in the two-sided case: \( \{\emptyset, \{F\}, \{T\}, \{T, F\}\} \).

Possible truth-values in the labelled case: \( P(L) \).

A valuation \( \nu \) is a model of a sequent \( \Omega \) if for some labelled formula \( a : \psi \) in \( \Omega \), \( a \in \nu(\psi) \).

**Primitive rules** determine the actual set of truth-values.

**Introduction rules** determine the truth-tables of the logical connectives.
Start with the calculus over $\mathcal{L} = \{a, b, c\}$ including only weakening.

$$\text{Vals} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$
Start with the calculus over $\mathcal{L} = \{a, b, c\}$ including only weakening.

$$\text{Vals} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$$

Now we add the primitive rules:

\[
\begin{align*}
\frac{s}{(\{a, b\} : \psi) \cup s} & \quad \frac{\{a : \psi\} \cup s}{s} \quad \frac{\{b : \psi\} \cup s}{s} \quad \frac{\{c : \psi\} \cup s}{s} \\
\text{Vals} = \{\{b\}, \{a\}, \{a, b\}\}
\end{align*}
\]
Example

Start with the calculus over \( L = \{a, b, c\} \) including only weakening.

\[
Vals = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}
\]

Now we add the primitive rules:

\[
\frac{s}{(\{a, b\} : \psi) \cup s}
\]

\[
\frac{\{a : \psi\} \cup s \quad \{b : \psi\} \cup s \quad \{c : \psi\} \cup s}{s}
\]

\[
Vals = \{\{b\}, \{a\}, \{a, b\}\}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th>(\tilde{\wedge})</th>
<th>{a}</th>
<th>{b}</th>
<th>{a, b}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
</tr>
<tr>
<td>{b}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
</tr>
<tr>
<td>{a, b}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
<td>{a}, {b}, {a, b}</td>
</tr>
</tbody>
</table>
Adding the introduction rule:

\[
\frac{\{a : \psi_1\} \cup s \quad \{a : \psi_2\} \cup s}{\{a : \psi_1 \land \psi_2\} \cup s}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th>$\land$</th>
<th>${a}$</th>
<th>${b}$</th>
<th>${a, b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
</tr>
<tr>
<td>${b}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
</tr>
<tr>
<td>${a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
<td>${a}, {b}, {a, b}$</td>
</tr>
</tbody>
</table>
Example

Adding the introduction rule:

\[
\begin{align*}
\{a : \psi_1\} \cup s & \quad \{a : \psi_2\} \cup s \\
\{a : \psi_1 \land \psi_2\} \cup s
\end{align*}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th></th>
<th>(\tilde{\land})</th>
<th>({a})</th>
<th>({b})</th>
<th>({a, b})</th>
</tr>
</thead>
<tbody>
<tr>
<td>({a})</td>
<td>{{a}, {a, b}}</td>
<td>{{a}, {b}, {a, b}}</td>
<td>{{a}, {a, b}}</td>
<td></td>
</tr>
<tr>
<td>({b})</td>
<td>{{a}, {b}, {a, b}}</td>
<td>{{a}, {b}, {a, b}}</td>
<td>{{a}, {b}, {a, b}}</td>
<td></td>
</tr>
<tr>
<td>({a, b})</td>
<td>{{a}, {a, b}}</td>
<td>{{a}, {b}, {a, b}}</td>
<td>{{a}, {a, b}}</td>
<td></td>
</tr>
</tbody>
</table>
Example

Adding the introduction rule:

\[
\frac{\{ b : \psi_1, b : \psi_2 \} \cup s}{\{ b : \psi_1 \land \psi_2 \} \cup s}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th>(\tilde{\land})</th>
<th>({a})</th>
<th>({b})</th>
<th>({a, b})</th>
</tr>
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<tbody>
<tr>
<td>({a})</td>
<td>({{a}, {a, b}})</td>
<td>({{a}, {b}, {a, b}})</td>
<td>({{a}, {a, b}})</td>
</tr>
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<td>({{a}, {b}, {a, b}})</td>
<td>({{a}, {b}, {a, b}})</td>
<td>({{a}, {b}, {a, b}})</td>
</tr>
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<td>({a, b})</td>
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<td>({{a}, {b}, {a, b}})</td>
<td>({{a}, {a, b}})</td>
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</tbody>
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Example

Adding the introduction rule:

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\frac{\{ b : \psi_1, b : \psi_2 \} \cup s}{\{ b : \psi_1 \land \psi_2 \} \cup s}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th>(\neg)</th>
<th>{a}</th>
<th>{b}</th>
<th>{a, b}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{{a}, {a, b}}</td>
<td>{{b}, {a, b}}</td>
<td>{{a, b}}</td>
</tr>
<tr>
<td>{b}</td>
<td>{{b}, {a, b}}</td>
<td>{{b}, {a, b}}</td>
<td>{{b}, {a, b}}</td>
</tr>
<tr>
<td>{a, b}</td>
<td>{{a, b}}</td>
<td>{{b}, {a, b}}</td>
<td>{{a, b}}</td>
</tr>
</tbody>
</table>
Adding the introduction rule:

\[
\frac{\{ b : \psi_1 \} \cup s \quad \{ b : \psi_2 \} \cup s}{\{ c : \psi_1 \land \psi_2 \} \cup s}
\]

The corresponding PNmatrix:

<table>
<thead>
<tr>
<th>( \wedge )</th>
<th>{a}</th>
<th>{b}</th>
<th>{a, b}</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{{a}, {a, b}}</td>
<td>{{b}, {a, b}}</td>
<td>{{a, b}}</td>
</tr>
<tr>
<td>{b}</td>
<td>{{b}, {a, b}}</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
<tr>
<td>{a, b}</td>
<td>{{a, b}}</td>
<td>\emptyset</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>
All Labelled Calculi are Decidable

Theorem

*Every canonical labelled calculus has a characteristic (finite) PNmatrix.*
All Labelled Calculi are Decidable

Theorem

Every canonical labelled calculus has a characteristic (finite) PNmatrix.

Corollary

Any logic induced by canonical labelled calculus is decidable.
### The Subformula Property

Whenever $\Omega \vdash_G s$, there is a derivation of $s$ from $\Omega$ in $G$ consisting solely of $E$-sequents (i.e. sequents consisting solely of formulas from $E$).
Application: characterization of syntactic properties

The Subformula Property
Whenever $\Omega \vdash_G s$, there is a derivation of $s$ from $\Omega$ in $G$ consisting solely of $\mathcal{E}$-sequents (i.e. sequents consisting solely of formulas from $\mathcal{E}$).

Strong Cut-Admissibility
Whenever $\Omega \vdash_G s$, there is a derivation of $s$ from $\Omega$ in $G$ in which cuts are allowed only on formulas from $\Omega$.

We call cut any primitive rule of the form $\frac{(L_1 : \psi) \ldots (L_n : \psi)}{s}$
The Subformula Property
Whenever $\Omega \vdash^G s$, there is a derivation of $s$ from $\Omega$ in $G$ consisting solely of $E$-sequents (i.e. sequents consisting solely of formulas from $E$).

Strong Cut-Admissibility
Whenever $\Omega \vdash^G s$, there is a derivation of $s$ from $\Omega$ in $G$ in which cuts are allowed only on formulas from $\Omega$. 

\[
\frac{(L_1 : \psi) \ldots (L_n : \psi)}{s}
\]

We call cut any primitive rule of the form $s$

Are these properties equivalent?
The subformula property \(\nRightarrow\) strong cut-admissibility

\[ \mathcal{L} = \{a, b, c\} \]

\(G\) has the following cuts:

\[
\begin{align*}
\{a : \psi\} & \cup s & \{b : \psi\} & \cup s & \{a : \psi\} & \cup s & \{c : \psi\} & \cup s & \{b : \psi\} & \cup s & \{c : \psi\} & \cup s \\
\end{align*}
\]

and the following introduction rules:

\[
\begin{align*}
(\{a, b\} : \psi) & \cup s & (\{b, c\} : \psi) & \cup s \\
\{a : \odot \psi\} & \cup s & \{a : \odot \psi\} & \cup s
\end{align*}
\]

Then we can derive:

\[
\begin{align*}
\{a : \psi\} & \\
\{a, b\} & : \odot \psi & \{a : \psi\} & \\
\{b, c\} & : \odot \psi & \{b : \odot \psi\} & \quad \text{cut}
\end{align*}
\]

But \(\{b : \odot \psi\}\) has no derivation from \(\{a : \psi\}\) with cuts only on \(\psi\).
The problem can be solved by adding the primitive rule (which does not affect the semantics of the calculus):

\[
\frac{(\{a, b\} : \psi) \cup s}{\{b : \psi\} \cup s} \quad \frac{(\{b, c\} : \psi) \cup s}{pr}
\]

Then we have a (cut-free!) derivation:

\[
\frac{\{a : \psi\}}{\{a, b\} : \star\psi} \quad \frac{\{a : \psi\}}{\{b, c\} : \star\psi} \quad \frac{\{b : \star\psi\}}{pr}
\]
The problem can be solved by adding the primitive rule (which does not affect the semantics of the calculus):

\[
\frac{\{a, b\} : \psi \cup s \quad \{b, c\} : \psi \cup s}{\{b : \psi\} \cup s \quad \text{pr}}
\]

Then we have a (cut-free!) derivation:

\[
\frac{\{a : \psi\}}{\{a, b\} : \star \psi \quad \text{pr}} \quad \frac{\{a : \psi\}}{\{b, c\} : \star \psi \quad \text{pr}}
\]

The addition of all such harmless primitive rules leads to a cut-saturated calculus.

**Theorem**

*For every labelled canonical calculus \( \mathbf{G} \) an equivalent cut-saturated \( \mathbf{G}' \) can be constructed.*
Finally: a semantic characterization

**Theorem**

Let $G$ be a cut-saturated canonical labelled calculus. Then the following statements are equivalent:

1. $G$ has a *proper* characteristic PNmatrix.
2. $G$ enjoys strong cut-admissibility.
3. $G$ enjoys the subformula property.
Our goals:
- Characterization of important syntactic properties of calculi.
- Understanding the dependencies between them.

Our tool: non-deterministic semantics.

Our case study: canonical labelled calculi.
The techniques can be applied to many families of proof systems: single-conclusioned canonical calculi, basic systems, canonical Gödel hypersequent systems and more.

Future research directions:
- First-order case
- Extension to calculi with less restrictive primitive and introduction rules.
- Substructural logics...