A Tutorial on Canonical Gentzen-type Systems

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What is a Propositional Logic?

Scott consequence relation (scr) between sets of formulas:

**strong reflexivity**: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.

**monotonicity**: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$ then $\Gamma' \vdash \Delta'$.

**transitivity** *(cut)*: if $\Gamma \vdash \psi$, $\Delta$ and $\Gamma, \psi \vdash \Delta$ then $\Gamma \vdash \Delta$.

Tarskian consequence relation (tcr) between sets of formulas and formulas:

**strong reflexivity**: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.

**monotonicity**: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.

**transitivity** *(cut)*: if $\Gamma \vdash \psi$ and $\Gamma, \psi \vdash \varphi$ then $\Gamma \vdash \varphi$. 
What is a Propositional Logic?

• A tcr or scr ⊢ for ℒ is structural if for every uniform ℒ-substitution σ and every Γ and Δ: if Γ ⊩ Δ then σ[Γ] ⊩ σ[Δ].

• A tcr or scr ⊢ for ℒ is consistent if there exist non-empty Γ and Δ, such that Γ ∉ Δ.

• A tcr or scr ⊢ for ℒ is finitary if whenever Γ ⊩ Δ, there exist finite Γ′ ⊆ Γ and Δ′ ⊆ Δ, such that Γ′ ⊩ Δ′.

• A propositional logic is a pair ⟨ℒ, ⊩⟩, where ⊩ is a tcr or scr for ℒ which is structural, consistent and finitary.
Gentzen-type Systems

- A Gentzen-type system $G$ is an axiomatic system which manipulates sequents of the form $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta$ are finite sets of formulas.

- A Gentzen-type system $G$ is called standard if:
  1. Its set of axioms includes the standard axioms:
     $$\psi \Rightarrow \psi$$
  2. It has among its rules the standard structural rules:
     permutation, contraction, weakening and cut.

- The associated scr: $\Gamma \vdash_G \Delta$ iff $\Gamma' \Rightarrow \Delta'$ is a theorem of $G$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. 
The System GCPL

\[
\begin{align*}
(\neg \Rightarrow) & \quad & \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg \varphi, \Gamma \Rightarrow \Delta} & \quad (\Rightarrow \neg) & \quad & \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \varphi} \\
(\supset \Rightarrow) & \quad & \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \supset \psi, \Gamma \Rightarrow \Delta} & \quad (\Rightarrow \supset) & \quad & \frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \supset \psi} \\
(\land \Rightarrow) & \quad & \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \land \psi \Rightarrow \Delta} & \quad (\Rightarrow \land) & \quad & \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \land \psi} \\
(\lor \Rightarrow) & \quad & \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \lor \psi \Rightarrow \Delta} & \quad (\Rightarrow \lor) & \quad & \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \lor \psi}
\end{align*}
\]
Useful Notions of Cut Elimination

- A Gentzen-type system $G$ admits cut-elimination if whenever a sequent is provable in $G$, it also has a cut-free proof in $G$.

- A cut is called a $\Theta$-cut if the cut formula occurs in $\Theta$.

- A cut is called $\Theta$-analytic if the cut formula is a subformula of some formula occurring in $\Theta$. A proof is $\Theta$-analytic if all cuts in it are $\Theta$-analytic.

- A Gentzen-type system $G$ admits strong cut-elimination if whenever a sequent is provable in $G$ from a set of sequents $\Theta$, it also has a proof in which all cuts are $\Theta$-cuts.

- $G$ admits strong analytic cut-elimination if whenever $\Theta \vdash_G \Omega$, $\Omega$ has in $G$ a $\Theta \cup \{\Omega\}$-analytic proof from $\Theta$. 
“Ideal” Logical Rules

- Gentzen’s vision of a “well-behaved” rule:

  “...The introductions represent, as it were, the ‘definitions’ of the symbols concerned...”

  G. Gentzen, “Investigations into Logical Deduction”.

- Thesis: the meaning of the connective is given by its introduction rules. *(Strongly challenged by Prior by using his famous “Tonk” connective...)*
What is a Canonical Rule?

- An “ideal” logical rule: an introduction rule or an elimination rule for \textit{exactly one connective}.

- In its formulation: \textit{exactly one occurrence} of the introduced connective, no other occurrences of other connectives.

- The rule should also be \textit{pure} (i.e. context-independent): no side conditions limiting its application.

- Its active formulas: \textit{immediate subformulas} of its principal formula.
What is a Canonical Rule?

\[
\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \land \varphi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \land \varphi}
\]

\[
\frac{\psi, \varphi \Rightarrow}{\psi \land \varphi \Rightarrow} \quad \frac{\Rightarrow \psi \Rightarrow \varphi}{\Rightarrow \psi \land \varphi}
\]

\[
\{p_1, p_2 \Rightarrow\}/p_1 \land p_2 \Rightarrow \quad \{\Rightarrow p_1 ; \Rightarrow p_2\}/ \Rightarrow p_1 \land p_2
\]
Canonical Systems: The Multiple-conclusion Case

- **A sequent**: an expression of the form \( \Gamma \Rightarrow \Delta \), where \( \Gamma \) and \( \Delta \) are finite sets of \( \mathcal{L} \)-formulas.

- **A clause**: a sequent consisting of atomic formulas.

- **A canonical rule** has one of the forms:

\[
\begin{align*}
\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} &/ \bigcirc (p_1, \ldots, p_n) \Rightarrow \\
\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m} &\Rightarrow \bigcirc (p_1, \ldots, p_n)
\end{align*}
\]

where \( m \geq 0 \) and for all \( 1 \leq i \leq m \): \( \Pi_i \Rightarrow \Sigma_i \) is a clause over \( \{p_1, \ldots, p_n\} \).
Canonical Rules

Application of a canonical rule of the form
\[ \{ \Pi_i \Rightarrow \Sigma_i \}_{1 \leq i \leq m} / \diamond (p_1, \ldots, p_n) \Rightarrow: \]

\[ \{ \Gamma, \Pi_i^* \Rightarrow \Delta, \Sigma_i^* \}_{1 \leq i \leq m} \]

\[ \frac{\Gamma, \diamond (\psi_1, \ldots, \psi_n) \Rightarrow \Delta}{\Gamma, \diamond (p_1, \ldots, p_n) \Rightarrow \Delta} \]

where \( \Pi_i^* \) and \( \Sigma_i^* \) are obtained from \( \Pi_i \) and \( \Sigma_i \) respectively by substituting \( \psi_j \) for \( p_j \) for all \( 1 \leq j \leq n \), and \( \Gamma, \Delta \) are any finite sets of formulas.
Example 1

Conjunction rules:

\[
\{p_1, p_2 \Rightarrow\} / p_1 \land p_2 \Rightarrow \{\Rightarrow p_1 ; \Rightarrow p_2\} / \Rightarrow p_1 \land p_2
\]

Their applications:

\[
\frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \land \varphi \Rightarrow \Delta}
\]

\[
\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \land \varphi}
\]
Example 2

Implication rules:

\[ \{ p_1 \Rightarrow p_2 \} / \Rightarrow p_1 \supset p_2 \quad \{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \supset p_2 \Rightarrow \]

Their applications:

\[ \frac{\Gamma, \psi \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \supset \varphi} \quad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta} \]
Example 3

Semi-implication rules:

\[ \{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \sim p_2 \Rightarrow \{ \Rightarrow p_2 \} / \Rightarrow p_1 \sim p_2 \]

Their applications:

\[ \Gamma \Rightarrow \Delta, \psi \quad \Gamma, \varphi \Rightarrow \Delta \]
\[ \Gamma, \psi \sim \varphi \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta, \varphi \]
\[ \Gamma \Rightarrow \Delta, \psi \sim \varphi \]
Example 4

“Tonk” rules:

\[
\{p_2 \Rightarrow\} / p_1 T p_2 \Rightarrow \quad \{\Rightarrow p_1\} / \Rightarrow p_1 T p_2
\]

Their applications:

\[
\frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi T \psi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \varphi T \psi}
\]
What Sets of Rules are Acceptable?

- A standard Gentzen-type system is canonical if each of its logical rules is canonical.
- If $G$ is a canonical calculus, then $\vdash_G$ is a structural and finitary scr. But is it a logic?
Coherence - an Example

\( \{ p_1, p_2 \Rightarrow \} / p_1 \land p_2 \Rightarrow \{ \Rightarrow p_1 ; \Rightarrow p_2 \} / \Rightarrow p_1 \land p_2 \)

These two classical rules form a coherent pair, where:

\[
S_1 = \{ p_1, p_2 \Rightarrow \} \quad S_2 = \{ \Rightarrow p_1 ; \Rightarrow p_2 \}
\]

\[
S_1 \cup S_2 = \{ p_1, p_2 \Rightarrow ; \Rightarrow p_1 ; \Rightarrow p_2 \}
\]

\( \{ \Rightarrow p_1 \} / p_1 \circ p_2 \Rightarrow \{ \Rightarrow p_1 ; \Rightarrow p_2 \} / \Rightarrow p_1 \circ p_2 \)

These two rules form an incoherent pair, where:

\[
S_1 = \{ \Rightarrow p_1 \} \quad S_2 = \{ \Rightarrow p_1 ; \Rightarrow p_2 \}
\]

\[
S_1 \cup S_2 = \{ \Rightarrow p_1 ; \Rightarrow p_2 \}
\]
A canonical calculus $G$ is **coherent** if for every pair of rules $\Theta_1/ \Rightarrow \Diamond(p_1, \ldots, p_n)$ and $\Theta_2/ \Diamond(p_1, \ldots, p_n) \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent (and so the empty set can be derived from it using cuts).

For a canonical calculus $G$, $\vdash_G$ is a logic iff $G$ is coherent.
Coherent Calculi:

\[ \{p_1, p_2 \Rightarrow \} / p_1 \land p_2 \Rightarrow \{ \Rightarrow p_1 ; \Rightarrow p_2 \} / \Rightarrow p_1 \land p_2 \]

\[ \{p_1 \Rightarrow p_2 \} / \Rightarrow p_1 \supset p_2 \quad \{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \supset p_2 \Rightarrow \]

\[ \{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \dashv p_2 \Rightarrow \{ \Rightarrow p_2 \} / \Rightarrow p_1 \dashv p_2 \]

\[ \{p_1 \Rightarrow \} / \Rightarrow \neg p_1 \quad \{ \Rightarrow p_1 \} / \neg p_1 \Rightarrow \]
Non-coherent: “Tonk”!

\{ p_2 \Rightarrow \} / p_1 T p_2 \Rightarrow \{ \Rightarrow p_1 \} / \Rightarrow p_1 T p_2

This is what is wrong with “Tonk”: these rules do not define a logic!
Splitting Canonical Rules

\[
\begin{align*}
\frac{p_1, p_2 \Rightarrow}{p_1 \land p_2 \Rightarrow} \\
\downarrow \\
\frac{p_1 \Rightarrow}{p_1 \land p_2 \Rightarrow} \quad \frac{p_2 \Rightarrow}{p_1 \land p_2 \Rightarrow}
\end{align*}
\]
“Reading off” the Semantics from Canonical Rules

\[
\begin{align*}
\Rightarrow \psi & \Rightarrow \varphi \\
\frac{}{\Rightarrow \psi \land \varphi} \\
\Rightarrow \psi & \varphi \Rightarrow \\
\frac{}{\psi \supset \varphi} \\
\Rightarrow \psi & \Rightarrow \varphi \\
\varphi \Rightarrow & \\
\frac{}{\Rightarrow \psi \land \varphi} \\
\Rightarrow \psi & \Rightarrow \\
\psi \land \varphi \Rightarrow \\
\frac{}{\Rightarrow \psi \supset \varphi}
\end{align*}
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How to Deal with Underspecification

\[
\frac{\Rightarrow \psi \quad \Rightarrow \varphi}{\Rightarrow \psi \land \varphi}
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\[
\frac{\varphi \Rightarrow}{\psi \land \varphi \Rightarrow}
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\frac{\Rightarrow \psi \quad \varphi \Rightarrow}{\Rightarrow \psi \supset \varphi}
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### Non-deterministic Matrices

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&\Rightarrow \psi \supset \varphi
\end{align*}
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Non-deterministic Matrices

\( \mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \) is a non-deterministic matrix (Nmatrix) for \( \mathcal{L} \) if:

- \( \mathcal{V} \) is a nonempty set of truth-values.
- \( \emptyset \neq \mathcal{D} \subset \mathcal{V} \) is a set of designated truth-values.
- For every \( n \)-ary connective \( \diamond \) of \( \mathcal{L} \), \( \mathcal{O} \) includes an operation \( \tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V}) \)

Any ordinary (deterministic) matrix can be identified with an Nmatrix whose functions in \( \mathcal{O} \) always return singletons.
Non-deterministic Matrices

Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$.

- An $\mathcal{M}$-valuation is a function $v$ from $\mathcal{L}$-formulas to $\mathcal{V}$, such that:

  $$v[\diamond(\psi_1 \ldots \psi_n)] \in \widetilde{\diamond}[v[\psi_1] \ldots v[\psi_n]]$$

- An $\mathcal{M}$-valuation $v$ satisfies a formula $\psi$ if $v[\psi] \in \mathcal{D}$. $v$ satisfies a set of formulas $\Gamma$ if it satisfies every formula in $\Gamma$.

- $\Gamma \vdash_{\mathcal{M}} \Delta$ if for every $\mathcal{M}$-valuation $v$ which satisfies $\Gamma$, $v$ satisfies some formula in $\Delta$.

- $\mathcal{M}$ is a characteristic Nmatrix for a calculus $G$ if $\vdash_G = \vdash_{\mathcal{M}}$. 
An obvious, yet crucial fact: any partial $\mathcal{M}$-legal assignment defined on a set closed under subformulas can be extended to a full $\mathcal{M}$-legal assignment.

If $\mathcal{M}$ is finite this entails that $\vdash_\mathcal{M}$ is:

- Decidable
- Finitary (the compactness theorem obtains)
Every Coherent Calculus Has a Characteristic 2Nmatrix

Consider the canonical calculus $G_0$ over the language \{\&, \\bowtie\} with no canonical rules whatsoever:

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Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{p_1, p_2 \Rightarrow\}/p_1 \land p_2 \Rightarrow$

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Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule \( \{ \Rightarrow p_2 \} / \Rightarrow p_1 \rightsquigarrow p_2 \)

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Every Coherent Calculus Has a Characteristic 2Nmatrix

Add the rule $\{\Rightarrow p_1 ; p_2 \Rightarrow\}/p_1 \sim p_2 \Rightarrow$

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Every 2Nmatrix Has a Corresponding Coherent Calculus

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$\{ \Rightarrow p_1 ; \Rightarrow p_2 \} / p_1 \circ p_2 \Rightarrow$

$\{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \circ p_2 \Rightarrow$

$\{ p_1 \Rightarrow ; p_2 \Rightarrow \} / \Rightarrow p_1 \circ p_2$

This is not the most efficient form - “normal form” to be defined.
Exact Correspondence

Let $G$ be a canonical calculus. The following statements concerning $G$ are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).
2. $G$ is coherent.
3. $G$ has a characteristic $2N$ matrix.
5. $G$ admits cut-elimination.

$1 \Rightarrow 2$ - see below. $2 \Rightarrow 3$ was demonstrated above. $3 \Rightarrow 4$ - see below. $4 \Rightarrow 5$ is trivial. $5 \Rightarrow 1$ - see below.
$\vdash G$ is consistent $\Rightarrow G$ is coherent

Suppose that there are two rules $\Theta_1 \vdash \Diamond(p_1, \ldots, p_n)$ and $\Theta_2 \vdash \Diamond(p_1, \ldots, p_n) \Rightarrow$, such that $\Theta_1 \cup \Theta_2$ is classically consistent. Then there is a classical valuation $v$ which satisfies $\Theta_1 \cup \Theta_2$. Let $\Pi' = \{p_i \mid 1 \leq i \leq n, v[p_i] = t\}$ and $\Sigma' = \{p_i \mid 1 \leq i \leq n, v[p_i] = f\}$. Let $\Theta'_j = \{\Pi, \Pi' \Rightarrow \Sigma, \Sigma' \mid \Pi \Rightarrow \Sigma \in \Theta_j\}$ for $j = 1, 2$. Then $\Theta'_1, \Theta'_2$ are sets of standard axioms. By applying the first rule on $\Theta'_1$ we obtain $\Pi', \Diamond(p_1, \ldots, p_n) \Rightarrow \Sigma'$. By applying the second rule on $\Theta'_2$ we obtain $\Pi' \Rightarrow \Sigma', \Diamond(p_1, \ldots, p_n)$. By cut we obtain a proof of $\Pi' \Rightarrow \Sigma'$. But then $p \Rightarrow q$ is provable for every $p \neq q$. Hence $\vdash G$ is not consistent.
G has a characteristic 2N matrix ⇒ G admits strong cut-elimination

Let $M_G$ be a characteristic 2N matrix for $G$. Suppose that $\Gamma \Rightarrow \Delta$ has no proper proof from $\Theta$ in $G$ (in which the only cuts are on formulas from $\Theta$). We show that $\Theta \nvdash M_G \Gamma \Rightarrow \Delta$. To define a refuting valuation, we first extend the sets $\Gamma, \Delta$ to sets $\Gamma^*, \Delta^*$, such that: (i) $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$, (ii) $\Gamma^* \Rightarrow \Delta^*$ has no proper proof in $G$ from $\Theta$, (iii) For every rule $\Theta/ \Rightarrow \Diamond(p_1, \ldots, p_n) (\Theta/ \Diamond(p_1, \ldots, p_n) \Rightarrow)$: whenever $\Diamond(\psi_1, \ldots, \psi_n) \in \Delta^*$ ($\Diamond(\psi_1, \ldots, \psi_n) \in \Gamma^*$), there is some $\Sigma \Rightarrow \Pi \in \Theta$ and some $1 \leq i \leq n$, such that either $p_i \in \Sigma$ and $\psi_i \in \Gamma^*$, or $p_i \in \Pi$ and $\psi_i \in \Delta^*$, and (iv) for every $\psi$ occurring in $\Theta$, $\psi \in \Gamma^* \cup \Delta^*$. Next a refuting valuation $v$ is defined, which satisfies the sequents in $\Theta$, but does not satisfy $\Gamma \Rightarrow \Delta$. The $M_G$-legality of $v$ is guaranteed by the properties above.
$G$ admits cut-elimination $\Rightarrow \vdash_G$ is consistent

Clauses which are not axioms can be proved only by cuts on atomic formulas. Thus if $G$ admits cut-elimination, it must be consistent.
Resolution Normal Form of Canonical Rules

• A canonical rule $S_1/C$ is at least as strong as the canonical rule $S_2/C$ iff every clause in $S_1$ classically follows from $S_2$. This is equivalent to saying that every clause in $S_1$ is subsumed by some clause that can be derived from the clauses of $S_2$ using resolutions.

• Two canonical rules $S_1/C$ and $S_2/C$ are equivalent if $S_1$ and $S_2$ are classically equivalent (as sets of clauses).

• A canonical rule is in Resolution Normal Form ($RNF$) if its set of premises $S$ does not include a standard axiom, and any resolvent of two elements of $S$ is subsumed by some other element of $S$.

• Every canonical rule has an equivalent canonical rule in $RNF$. 
Example

\[
\{ p_1 \Rightarrow p_2 ; p_2 \Rightarrow p_1 ; \Rightarrow p_1 , p_2 \} / \Rightarrow p_1 \land p_2
\]

An equivalent rule in RNF:

\[
\{ \Rightarrow p_1 ; \Rightarrow p_2 \} / \Rightarrow p_1 \land p_2
\]
Canonical Calculi in Normal Form

- A canonical calculus $G$ is in *normal form* if all the rules of $G$ are in RNF, and for each connective $G$ has at most one left introduction rule and at most one right introduction rule.

- Any canonical calculus can be transformed into a cut-free equivalent normal form.
The Syntactic Method

Consider the following two rules for the binary connective $X$ (representing XOR):

$$\{ \Rightarrow p_1 ; p_2 \Rightarrow \} / \Rightarrow p_1 X p_2 \quad \{ \Rightarrow p_2 ; p_1 \Rightarrow \} / \Rightarrow p_1 X p_2$$

\[
\downarrow
\]

$$\{ \Rightarrow p_1, p_2 ; p_1 \Rightarrow p_1 ; p_2 \Rightarrow p_2 ; p_1, p_2 \Rightarrow \} / \Rightarrow p_1 X p_2$$

\[
\downarrow
\]

$$\{ \Rightarrow p_1, p_2 ; p_1, p_2 \Rightarrow \} / \Rightarrow p_1 X p_2$$
The Semantic Method

If a characteristic 2Nmatrix $\mathcal{M}$ is given for a calculus, its equivalent normal form can be easily constructed as follows:

$(\Diamond \Rightarrow) :$

$$\{ \{ p_i \mid x_i = t \} \Rightarrow \{ p_i \mid x_i = f \} \mid t \in \tilde{\mathcal{M}}[x_1, \ldots, x_n] \}$$

$$\Diamond(p_1, \ldots, p_n) \Rightarrow$$

$(\Rightarrow \Diamond) :$

$$\{ \{ p_i \mid x_i = t \} \Rightarrow \{ p_i \mid x_i = f \} \mid f \in \tilde{\mathcal{M}}[x_1, \ldots, x_n] \}$$

$$\Rightarrow \Diamond(p_1, \ldots, p_n)$$

If $t \in \tilde{\mathcal{M}}[x_1, \ldots, x_n]$ for all $x_1, \ldots, x_n$, then the first rule is redundant and can be discarded.

If $\tilde{\mathcal{M}}[x_1, \ldots, x_n] = \{ f \}$ for all $x_1, \ldots, x_n$ then the first rule does not have any premises (a nonstandard axiom).
Example: The XOR Connective

<table>
<thead>
<tr>
<th>$p_1$</th>
<th>$p_2$</th>
<th>$p_1 \text{X} p_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$t$</td>
<td>${f}$</td>
</tr>
<tr>
<td>$t$</td>
<td>$f$</td>
<td>${t}$</td>
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<td>$f$</td>
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</tr>
<tr>
<td>$f$</td>
<td>$f$</td>
<td>${f}$</td>
</tr>
</tbody>
</table>

\[
\{ \Rightarrow p_1, p_2; p_1, p_2 \Rightarrow \} / \Rightarrow p_1 \text{X} p_2
\]

\[
\{ p_1 \Rightarrow p_2; p_2 \Rightarrow p_1 \} / p_1 \text{X} p_2 \Rightarrow
\]
**When are Canonical Rules Invertible?**

- A rule $R$ is *invertible in a calculus $G$* if for every application of $R$ it holds that whenever its conclusion is provable in $G$, also each of its premises is provable in $G$.

- Calculus $G_1$ (the first rule is invertible):

  $$\{p_1, p_2 \Rightarrow\}/p_1 \land p_2 \Rightarrow \{\Rightarrow p_1 ; \Rightarrow p_2\}/ \Rightarrow p_1 \land p_2$$

  Application of the first rule:

  $$\Gamma, \psi_1, \psi_2 \Rightarrow \Delta$$
  $$\Gamma \Rightarrow \psi_1 \land \psi_2, \Delta$$

  Invertibility of the first rule:

  $$\Gamma, \psi_1 \Rightarrow \Delta, \psi_1 \quad \Gamma, \psi_2 \Rightarrow \Delta, \psi_2$$
  $$\Gamma, \psi_1 \land \psi_2 \Rightarrow \Delta \quad \Gamma, \psi_1, \psi_2 \Rightarrow \Delta, \psi_1 \land \psi_2$$
  $$\Gamma, \psi_1, \psi_2 \Rightarrow \Delta$$
When are Canonical Rules Invertible?

- Calculus $G_1$:

  \[
  \{ p_1, p_2 \Rightarrow \} / p_1 \land p_2 \Rightarrow \{ \Rightarrow p_1 \ ; \Rightarrow p_2 \} / \Rightarrow p_1 \land p_2
  \]

- An (equivalent) calculus $G_2$:

  \[
  \{ p_1 \Rightarrow \} / p_1 \land p_2 \Rightarrow \{ p_2 \Rightarrow \} / p_1 \land p_2 \Rightarrow \{ \Rightarrow p_1 \ ; \Rightarrow p_2 \} / \Rightarrow p_1 \land p_2
  \]

  The first rules are NOT invertible: $\vdash_{G_2} \psi_1 \land \psi_2 \Rightarrow \psi_1$, but $\forall_{G_2} \psi_1 \Rightarrow \psi_2$.

  *The explanation: $G_2$ is not in normal form.*
Invertibility and Determinism

Let $G$ be a coherent canonical calculus in normal form. The following statements are equivalent:

1. $G$ has an invertible rule for $\Diamond$.
2. $\hat{\Diamond}_{\mathcal{M}_G}$ (constructed as explained above) is deterministic.
3. $G$ has a rule for $\Diamond$, and all the rules for $\Diamond$ are invertible.
Axiom Expansion

- Axiom expansion in a calculus allows for a reduction of logical axioms to the atomic case.

- An $n$-ary connective $\Diamond$ **admits axiom expansion** in a calculus $G$ if whenever $\Diamond(p_1, \ldots, p_n) \Rightarrow \Diamond(p_1, \ldots, p_n)$ is provable in $G$, it has a cut-free derivation in $G$ from atomic axioms of the form $\{p_i \Rightarrow p_i\}_{1 \leq i \leq n}$. 
Another Exact Correspondence

Let $G$ be a coherent canonical calculus in normal form. The following statements are equivalent:

1. The rules of $G$ are invertible,
2. $G$ has a characteristic two-valued deterministic matrix.
3. Every connective of $\mathcal{L}$ admits axiom expansion in $G$. 
Canonical Calculi: The Single-conclusion Case

- **A positive Horn clause:** a sequent of the form $\Pi \Rightarrow \{q\}$, where $\Pi$ is a set of atomic formulas and $q$ - an atomic formula.

- **A negative Horn clause:** a sequent of the form $\Pi \Rightarrow$, where $\Pi$ is a set of atomic formulas.

- **A single-conclusioned sequent:** an expression $\Gamma \Rightarrow \psi$, where $\Gamma$ is a set of $\mathcal{L}$-formulas and $\psi$ - an $\mathcal{L}$-formula.
Canonical Single-conclusion Rules

- A canonical introduction rule:

$$\{ \Pi_i \Rightarrow \Sigma_i \}_{1 \leq i \leq m} \Rightarrow \diamond (p_1, \ldots, p_n)$$

where \( m \geq 0 \) and for all \( 1 \leq i \leq m \): \( \Pi_i \Rightarrow \Sigma_i \) is a positive Horn clause over \( \{ p_1, \ldots, p_n \} \).

- A canonical elimination rule:

$$\{ \Pi_i \Rightarrow \Sigma_i \}_{1 \leq i \leq m} \Rightarrow \diamond (p_1, \ldots, p_n) \Rightarrow$$

where \( m \geq 0 \) and for all \( 1 \leq i \leq m \): \( \Pi_i \Rightarrow \Sigma_i \) is a Horn clause (either positive or negative) over \( \{ p_1, \ldots, p_n \} \).
Applications of Rules

Application of \( \{\Pi_i \Rightarrow \Sigma_i\} \) \( 1 \leq i \leq m \) / \( \Diamond(p_1, \ldots, p_n) \Rightarrow \):

\[
\frac{\{\Gamma, \Pi^*_i \Rightarrow \varphi_i\} \quad 1 \leq i \leq m}{\Gamma, \Diamond(\psi_1, \ldots, \psi_n) \Rightarrow \theta}
\]

where \( \Pi^*_i \) is obtained from \( \Pi_i \) by substituting \( \psi_j \) for \( p_j \) for all \( 1 \leq j \leq n \), \( \varphi_i = \psi_j \) in case \( \Sigma_i = \{p_j\} \), \( \varphi_i = \theta \) in case \( \Sigma_i \) is empty, and \( \Gamma \) is any finite set of formulas.
Example 1

Conjunction rules:

\[
\{ p_1, p_2 \implies \} / p_1 \land p_2 \implies \{ \implies p_1 ; \implies p_2 \} / \implies p_1 \land p_2
\]

Their applications:

\[
\begin{align*}
\Gamma, \psi, \varphi & \implies \theta \\
\Gamma, \psi \land \varphi & \implies \theta \\
\Gamma & \implies \psi \\
\Gamma & \implies \varphi \\
\Gamma & \implies \psi \land \varphi
\end{align*}
\]
Example 2

Implication rules:

\[
\begin{align*}
\{p_1 \Rightarrow p_2\} & / \Rightarrow p_1 \supset p_2 & \{\Rightarrow p_1 ; p_2 \Rightarrow\} & / p_1 \supset p_2 \Rightarrow
\end{align*}
\]

Their applications:

\[
\begin{align*}
\Gamma, \psi \Rightarrow \varphi & \\
\hline
\Gamma & \Rightarrow \psi \supset \varphi
\end{align*}
\]

\[
\begin{align*}
\Gamma & \Rightarrow \psi \\
\hline
\Gamma, \varphi \Rightarrow \theta
\end{align*}
\]

\[
\begin{align*}
\Gamma, \psi \supset \varphi & \Rightarrow \theta
\end{align*}
\]
Example 3

Semi-implication rules:

\[ \{ \Rightarrow \; p_1 \; ; \; p_2 \; \Rightarrow \} \; / \; p_1 \; \leadsto \; p_2 \; \Rightarrow \; \{ \Rightarrow \; p_2 \} \; / \; \Rightarrow \; p_1 \; \leadsto \; p_2 \]

Their applications:

\[
\frac{\Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta}{\Gamma, \psi \leadsto \varphi \Rightarrow \theta}
\]

\[
\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \leadsto \varphi}
\]
Yet Another Exact Correspondence

Let $G$ be a single-conclusioned canonical calculus. The following statements concerning $G$ are equivalent:

1. $\vdash_G$ is consistent (and so it is a logic).

2. $G$ is coherent.


Signed Formulas: the Two-valued Case

A version of Gentzen-type calculi using signed formulas:

\[
\begin{align*}
\psi, \varphi & \Rightarrow \\
\psi \land \varphi & \Rightarrow
\end{align*}
\]

\[
\Rightarrow \psi \Rightarrow \varphi \Rightarrow \psi \land \varphi
\]

\[
\begin{align*}
\{f : \psi, f : \varphi\} & \\
\{f\} : \psi \land \varphi
\end{align*}
\]

\[
\begin{align*}
\{t : \psi\} \quad \{t : \varphi\} & \\
\{t\} : \psi \land \varphi
\end{align*}
\]
Signed Formulas: the General Case

Let $\mathcal{V}$ be a finite set of signs.

- A signed formula: an expression of the form $a : \psi$, where $\psi$ is a formula and $a \in \mathcal{V}$.
- A sequent: a finite set of signed formulas.
- A clause: a sequent consisting of atomic signed formulas.
- A valuation $v$ satisfies a signed formula $a : \psi$ if $v[\psi] = a$.
- $v$ satisfies a set of signed formulas $\Omega$, if it satisfies some signed formula in $\Omega$. Sequents are interpreted disjunctively.
- $v$ satisfies a set of sequents $\Theta$, if it satisfies all sequents of $\Theta$. 
• A logical axiom for $\mathcal{V}$ is a sequent of the form $\{l : \psi \mid l \in \mathcal{V}\}$.

• The cut and weakening rules for $\mathcal{V}$ are defined as follows:

$$
\begin{align*}
\Omega \cup \{l : \psi \mid l \in L_1\} & \quad \Omega \cup \{l : \psi \mid l \in L_2\} \\
\hline
\Omega \cup \{l : \psi \mid l \in L_1 \cap L_2\}
\end{align*}
$$

where $L_1, L_2 \subseteq \mathcal{V}$

$$
\frac{\Omega}{\Omega, l : \psi}
$$

where $l \in \mathcal{V}$. 
Canonical Signed Rules

- $\mathcal{V} = \{t, f\}$:
  
  $\{p_1 \Rightarrow p_2\} / \Rightarrow p_1 \supset p_2$
  
  $\{\Rightarrow p_1 , p_2 \Rightarrow\} / p_1 \supset p_2 \Rightarrow$
  
  $\{\{f : p_1, t : p_2\}\} / \{t\} : p_1 \supset p_2$
  
  $\{\{t : p_1\} , \{ f : p_2\}\} / \{f\} : p_1 \supset p_2$

- $\mathcal{V} = \{a, b, c\}$:
  
  $\{\{a : p_1, c : p_2\} , \{a : p_3, b : p_2\}\} / \{a, c\} : \circ(p_1, p_2, p_3)$
  
  $\{\{c : p_2\} , \{a : p_3, b : p_3\} , \{c : p_1\}\} / \{b, c\} : \circ(p_1, p_2, p_3)$
Canonical Signed Rules

• A signed canonical rule for an \( n \)-ary connective \( \Diamond \):

\[
\{ \Sigma_1, ..., \Sigma_m \} / S : \Diamond(p_1, \ldots, p_n)
\]

where \( S \subset V \), \( m \geq 0 \) and for every \( 1 \leq j \leq m \): \( \Sigma_j \) is a clause consisting of atomic signed formulas of the form \( a : p_k \) for \( a \in V \) and \( 1 \leq k \leq n \).

• An application of a rule \( \{ \Sigma_1, ..., \Sigma_m \} / S : \Diamond(p_1, \ldots, p_n) \):

\[
\frac{\Omega \cup \Sigma_i^* \quad \ldots \quad \Omega \cup \Sigma_m^*}{\Omega \cup S : \Diamond(\psi_1, \ldots, \psi_n)}
\]

where \( \psi_1, ..., \psi_n \) are \( L \)-formulas, \( \Omega \) is a sequent, and for all \( 1 \leq i \leq m \): \( \Sigma_i^* \) is obtained from \( \Sigma_i \) by replacing \( p_j \) by \( \psi_j \) for every \( 1 \leq j \leq n \).
Example 1

Standard conjunction rules:

\[
\begin{align*}
\{ \{ f : p_1, f : p_2 \} \} & / \{ f \} : p_1 \land p_2 \quad \{ \{ t : p_1 \}, \{ t : p_2 \} \} & / \{ t \} : p_1 \land p_2
\end{align*}
\]

Their applications:

\[
\begin{align*}
\Omega \cup \{ f : \psi_1, f : \psi_2 \} & \quad \Rightarrow \quad \Omega \cup \{ f : \psi_1 \land \psi_2 \} \\
\Omega \cup \{ t : \psi_1 \} & \quad \Rightarrow \quad \Omega \cup \{ t : \psi_1 \land \psi_2 \} \\
\Omega \cup \{ t : \psi_2 \} & \quad \Rightarrow \quad \Omega \cup \{ t : \psi_1 \land \psi_2 \}
\end{align*}
\]
Example 2

Rule for a treenary connective $\circ$:

$$\{\{a : p_1, c : p_2\}, \{a : p_3, b : p_2\}\} / \{a, c\} : \circ(p_1, p_2, p_3)$$

Its application:

$$\Omega \cup \{a : \psi_1, c : \psi_2\} \quad \Omega \cup \{a : \psi_3, b : \psi_2\}$$

$$\Omega \cup \{a : \circ(\psi_1, \psi_2, \psi_3), c : \circ(\psi_1, \psi_2, \psi_3)\}$$
Coherence

- Example for $\mathcal{V} = \{a, b, c\}$:

\[
\begin{align*}
\{\{a : p_1\}, \{c : p_2\}\} &/ \{a, b\} : p_1 \diamond p_2 \\
\{\{a : p_1\}, \{c : p_2\}\} &/ \{b, c\} : p_1 \diamond p_2 \\
\{\{a : p_1\}, \{c : p_2\}\} &/ \{a, c\} : p_1 \diamond p_2
\end{align*}
\]

It is not enough to check only pairs of rules.

- A canonical signed calculus $G$ is coherent if $\Theta_1 \cup \ldots \cup \Theta_m$ is inconsistent (i.e., the empty sequent can be derived by cuts) whenever $\Theta_1/S_1 : \diamond (p_1, \ldots, p_n), \ldots, \Theta_m/S_m : \diamond (p_1, \ldots, p_n)$ is a set of rules of $G$, such that $S_1 \cap \ldots \cap S_m = \emptyset$. 
Examples

Coherent:

\[
\left\{ \{ f : p_1, f : p_2 \} \right\} / \{ f \} : p_1 \land p_2 \quad \{ \{ t : p_1 \}, \{ t : p_2 \} \} / \{ t \} : p_1 \land p_2
\]

\[
\begin{array}{c}
\{ t : p_1 \} \\
\{ f : p_1, f : p_2 \} \\
\{ f : p_2 \}
\end{array}
\]

\[
\frac{\text{cut}}{\{ t : p_2 \}}
\]

\[
\begin{array}{c}
\text{cut} \\
\emptyset
\end{array}
\]

Non-coherent:

\[
\left\{ \{ a : p_1 \}, \{ b : p_2 \} \right\} / \{ a, b \} : \diamond (p_1, p_2, p_3)
\]

\[
\left\{ \{ a : p_2, c : p_3 \} \right\} / \{ c \} : \diamond (p_1, p_2, p_3)
\]

The set \( \left\{ \{ a : p_1 \}, \{ b : p_2 \}, \{ a : p_2, c : p_3 \} \right\} \) is satisfiable (and thus consistent).
Semantics for Canonical Signed Calculi

- We use Nmatrices of the form $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for canonical calculi over a set of signs $\mathcal{V}$.

- For a set of sets of signed formulas $\Theta$ and a set of signed formulas $\Omega$: $\Theta \vdash_{\mathcal{M}} \Omega$ if whenever an $\mathcal{M}$-legal valuation satisfies $\Theta$, it satisfies some signed formula of $\Omega$.

- The connection: $\Gamma \vdash_{\mathcal{M}} \Delta$ iff
  \[ \vdash_{\mathcal{M}} \left\{ \mathcal{D} : \psi \mid \psi \in \Delta \right\} \cup \left\{ \mathcal{V} - \mathcal{D} : \psi \mid \psi \in \Gamma \right\} \]
“Reading off” the Semantics from Canonical Rules

Let $\mathcal{V} = \{t, f, \top, \bot\}$. Let $G$ be a calculus with no rules for $\circ$.

\[
\begin{array}{|c|c|c|c|c|}
\hline
\circ & t & f & \top & \bot \\
\hline
 t & \checkmark & \checkmark & \checkmark & \checkmark \\
 f & \checkmark & \checkmark & \checkmark & \checkmark \\
 \top & \checkmark & \checkmark & \checkmark & \checkmark \\
 \bot & \checkmark & \checkmark & \checkmark & \checkmark \\
\hline
\end{array}
\]
“Reading off” the Semantics from Canonical Rules

Add the rule

\[ \{ \{ f : p_1, f : p_2 \} \} / \{ \bot, f \} : p_1 \circ p_2 \]

<table>
<thead>
<tr>
<th></th>
<th>t</th>
<th>f</th>
<th>T</th>
<th>\bot</th>
</tr>
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<tbody>
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<td>T</td>
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<td>{ \bot, f }</td>
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<td>\bot</td>
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<td>{ \bot, f }</td>
<td>\top</td>
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</tr>
</tbody>
</table>
“Reading off” the Semantics from Canonical Rules

Add the rule

\[ \{\{t : p_1, \top : p_1\}\} / \{f\} : p_1 \circ p_2 \]

<table>
<thead>
<tr>
<th></th>
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<th>(t)</th>
<th>(f)</th>
<th>(\top)</th>
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<td>(\forall)</td>
<td>{(\bot, f)}</td>
<td>(\forall)</td>
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</tr>
</tbody>
</table>
Another Exact Correspondence

Let $G$ be a canonical calculus. The following statements concerning $G$ are equivalent.

1. $G$ is coherent.

2. $G$ has a strongly characteristic Nmatrix.


What about (Strong) Cut-elimination?

\[
\{\{a : p_1\}\} / \{b, c\} : p_1 \circ p_2 \quad \{\{a : p_1\}\} / \{a, b\} : p_1 \circ p_2
\]

This calculus is coherent.

The sequent \(\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}\) has an analytic proof:

\[
\begin{align*}
\{a : p_1, b : p_1, c : p_1\} & \\
\{b : p_1, c : p_1, b : (p_1 \circ p_2), c : (p_1 \circ p_2)\} & \\
\{a : p_1, b : p_1, c : p_1\} & \\
\{b : p_1, c : p_1, a : (p_1 \circ p_2), b : (p_1 \circ p_2)\} & \\
\{b : p_1, c : p_1, b : (p_1 \circ p_2)\}
\end{align*}
\]

However, it has no cut-free proof.

*Coherence does not imply strong cut-elimination. However, by adding the rule \(\{\{a : p_1\}\} / \{b\} : p_1 \circ p_2\) strong cut-elimination is guaranteed.*
A canonical signed calculus $G$ is \textit{dense} if for every two rules of $G \Theta_1/S_1 : \diamond(p_1, \ldots, p_n)$ and $\Theta_2/S_2 : \diamond(p_1, \ldots, p_n)$ in $G$ there is some rule $\Theta/S : \diamond(p_1, \ldots, p_n)$, such that $\Theta \subseteq \Theta_1 \cup \Theta_2$ and $S \subseteq S_1 \cap S_2$.

Density implies coherence.

The following calculus is not dense:

$$\{\{a : p_1\}\} / \{a, b\} : p_1 \circ p_2 \quad \{\{a : p_1\}\} / \{b, c\} : p_1 \circ p_2$$

\textit{Correction: add $\{\{a : p_1\}\} / \{b\} : p_1 \circ p_2$, obtaining an equivalent calculus.}
Density Characterizes Strong Cut-elimination

Let $G$ be a canonical calculus. The following statements concerning $G$ are equivalent:

1. $G$ is dense.
2. $G$ admits cut-elimination.
Quantifiers: the First-order Case and Beyond

- A unary quantifier:
  \[ \forall x \psi \quad \exists x \psi \]

- An \( n \)-ary quantifier: \( Q_x(\psi_1, \ldots, \psi_n) \)
  \[
  \forall x (P(x), Q(x)) \equiv \forall x (P(x) \rightarrow Q(x))
  \]
  \[
  \exists x (P(x), Q(x)) \equiv \exists x (P(x) \land Q(x))
  \]
Universal quantification rules:

\[
\frac{\Gamma, A\{t/w\} \Rightarrow \Delta}{\Gamma, \forall w A \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w A, \Delta}
\]

where \( z \) is free for \( w \) in \( A \), \( z \) is not free in \( \Gamma \cup \Delta \cup \{\forall w A\} \), and \( t \) is any term free for \( w \) in \( A \).

\[
\frac{\frac{A\{t/w\} \Rightarrow \forall w A}{\Rightarrow \forall w A}}{\Rightarrow \forall w A}
\]

\[
\frac{\frac{\{p(c) \Rightarrow\} / \forall w p(w) \Rightarrow \quad \{\Rightarrow p(y)\} / \Rightarrow \forall w p(w)}{\Rightarrow \forall w p(w)}}
\]

Important: is an eigenvariable or a term used? Signify eigenvariable by a variable, and a term by a constant.
\( L^n \) - a Representation Language for Canonical Rules

- Represent \( Qx(\psi_1, ..., \psi_n) \) by \( Qx(p_1(x), ..., p_n(x)) \).

- For an \( n \)-ary introduction rule, \( L^n \) has \( n \) unary predicate symbols \( p_1, ..., p_n \) and a set of constants (no logical connectives).

- Represent the case of a term-variable \( (t) \) by a constant, and the case of an eigenvariable \( (y) \) by a variable.
A canonical rule of arity $n$ is an expression of one of the forms:

\[
\frac{\{\Pi_i \Rightarrow \Sigma_i\}_{1 \leq i \leq m}}{\Rightarrow Qx(p_1(x), \ldots, p_n(x))}
\]

where $Q$ is an $n$-ary quantifier, $m \geq 0$ and for every $1 \leq i \leq m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause over $L^n$. 
Instantiating Canonical Rules

Let $R = \Theta/ \Rightarrow Qx(p_1(x), \ldots, p_n(x))$ be an $n$-ary canonical rule, $\Gamma$ - some context of $L$-formulas, and $z$ a variable of $L$. A $\langle R, \Gamma, z \rangle$-mapping is any function $\mathcal{F}$ from the predicate symbols and terms of $L^n$ to formulas and terms of $L$, satisfying:

- For every $1 \leq i \leq n$, $\mathcal{F}[p_i]$ is an $L$-formula, $\mathcal{F}[y]$ is a variable of $L$ and $\mathcal{F}[c]$ is an $L$-term.
- $\mathcal{F}[x] \neq \mathcal{F}[y]$ for every two variables $x \neq y$ of $L^n$.
- For every $1 \leq i \leq n$ and every $p_i(t)$ occurring in $\Theta$:
  - $\mathcal{F}[t]$ is a term free for $z$ in $\mathcal{F}[p_i]$.
  - if $t$ is a variable, then $\mathcal{F}[t]$ does not occur free in $\Gamma \cup \{Qz(\mathcal{F}[p_1], \ldots, \mathcal{F}[p_n])\}$.
- $\mathcal{F}[p_i(t)] = \mathcal{F}[p_i]\{\mathcal{F}[t]/z\}$.

$\mathcal{F}$ is extended to sets of formulas: $\mathcal{F}(\Gamma) = \{\mathcal{F}(\psi) \mid \psi \in \Gamma\}$. 
Applications of Canonical Rules

An application of $R = \{\Sigma_j \Rightarrow \Pi_j\}_{1 \leq j \leq m} / \Rightarrow Qx(p_1(x), \ldots, p_n(x))$:

$$\frac{\{\Gamma, \mathcal{F}(\Sigma_j) \Rightarrow \Delta, \mathcal{F}(\Pi_j)\}_{1 \leq j \leq m}}{\Gamma \Rightarrow \Delta, Qz(\mathcal{F}(p_1), \ldots, \mathcal{F}(p_n))}$$

where $\mathcal{F}$ is some $\langle R, \Gamma \cup \Delta, z \rangle$-mapping.
Example 1

The two standard introduction rules for the unary quantifier $\forall$ can be formulated as follows:

$$\{p(c) \Rightarrow\} / \forall x p(x) \Rightarrow \{\Rightarrow p(y)\} / \Rightarrow \forall x p(x)$$

Applications of these rules have the forms:

$$\frac{\Gamma, A\{t/w\} \Rightarrow \Delta}{\Gamma, \forall w A \Rightarrow \Delta}$$
$$\frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w A, \Delta}$$

where $z$ is free for $w$ in $A$, $z$ is not free in $\Gamma \cup \Delta \cup \{\forall w A\}$, and $t$ is any term free for $w$ in $A$. 
Example 2

Consider the bounded universal binary quantifier \( \bar{\forall} \) (corresponding to \( \forall x(p_1(x) \to p_2(x)) \)).

\[
\{ p_2(c) \Rightarrow ; \Rightarrow p_1(c) \} / \bar{\forall} x \ (p_1(x), p_2(x)) \Rightarrow
\]
\[
\{ p_1(y) \Rightarrow p_2(y) \} / \Rightarrow \bar{\forall} x \ (p_1(x), p_2(x))
\]

Applications of these rules are of the form:

\[
\Gamma, \psi_2 \{ t/z \} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_1 \{ t/z \}, \Delta \quad \Gamma, \psi_1 \{ w/z \} \Rightarrow \psi_2 \{ w/z \}, \Delta
\]
\[
\Gamma, \bar{\forall} z \ (\psi_1, \psi_2) \Rightarrow \Delta \quad \Gamma \Rightarrow \bar{\forall} z \ (\psi_1, \psi_2), \Delta
\]

where \( t \) and \( w \) are free for \( z \) in \( \psi_1 \) and \( \psi_2 \), and \( w \) does not occur free in \( \Gamma \cup \Delta \cup \{ \bar{\forall} z(\psi_1, \psi_2) \} \).
Coherence

• A canonical calculus $G$ is coherent if for every two canonical rules of $G$ of the form $\Theta_1/ \Rightarrow A$ and $\Theta_2/ A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.

• The coherence of a canonical calculus $G$ is decidable.

• Examples: Coherent:

\[
\{ p(c) \Rightarrow \} / \forall x p(x) \Rightarrow \{ \Rightarrow p(y) \} / \Rightarrow \forall x p(x)
\]

Non-coherent:

\[
\{ \Rightarrow p(c) \} / \Rightarrow Qxp(x) \quad \{ p(d) \Rightarrow \} / Qxp(x) \Rightarrow
\]
Canonical Calculi with Quantifiers and $\alpha$-Equivalence

\[
\Gamma, A\{t/w\} \Rightarrow \Delta \quad \Gamma \Rightarrow A\{z/w\}, \Delta
\]

\[
\Gamma, \forall w A \Rightarrow \Delta \quad \Gamma \Rightarrow \forall w A, \Delta
\]

$\forall x P(x) \Rightarrow \forall y P(y)$ is derivable:

\[
P(y) \Rightarrow P(y)
\]

\[
\forall x P(x) \Rightarrow P(y)
\]

\[
\forall x P(x) \Rightarrow \forall y P(y)
\]
After discarding one rule for $\forall$, $\forall x P(x) \Rightarrow \forall y P(y)$ is no longer derivable.

Solution: add to canonical calculi an explicit axiom capturing the $\alpha$-equivalence principle:

$$A \Rightarrow A' \text{ if } A \equiv_\alpha A'$$
A substitution instance of $\Gamma \Rightarrow \Delta$ is any sequent of the form $\Gamma\{t_1/x_1, \ldots, t_n/x_n\} \Rightarrow \Delta\{t_1/x_1, \ldots, t_n/x_n\}$, where $t_i$ is free for $x_i$ in all the formulas of $\Gamma \cup \Delta$.

The substitution rule:

\[
\frac{\Gamma \Rightarrow \Delta}{\Gamma' \Rightarrow \Delta'} \quad \text{Sub}
\]

where $\Gamma' \Rightarrow \Delta'$ is a substitution instance of $\Gamma \Rightarrow \Delta$.

A Gentzen-type calculus $G$ with quantifiers is canonical if in addition to the $\alpha$-axiom, the substitution rule and the standard structural rules, $G$ has only canonical rules.
Unary Quantifiers in Deterministic Matrices

- A unary quantifier $Q$ is usually interpreted as a function $\tilde{Q} : P^+(\mathcal{V}) \to \mathcal{V}$.

- Examples:

  \[
  \begin{array}{c|c}
  H & \tilde{\forall}[H] \\
  \hline
  \{t\} & t \\
  \{t,f\} & f \\
  \{f\} & f \\
  \end{array}
  \quad
  \begin{array}{c|c}
  H & \tilde{\exists}[H] \\
  \hline
  \{t\} & t \\
  \{t,f\} & t \\
  \{f\} & f \\
  \end{array}
  \]

- A natural generalization to Nmatrices: $\tilde{Q} : P^+(\mathcal{V}) \to P^+(\mathcal{V})$. 
Nmatrices with Unary Quantifiers

\( \mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \) is a non-deterministic matrix (Nmatrix) for a language \( L \) with unary quantifiers if:

1. \( \mathcal{V} \) is a nonempty set

2. \( \emptyset \neq \mathcal{D} \subset \mathcal{V} \)

3. for every \( n \)-ary connective \( \diamond \) of \( L \), \( \mathcal{O} \) includes an operation
\[ \tilde{\diamond} : \mathcal{V}^n \rightarrow P^+(\mathcal{V}) \]

4. for every unary quantifier \( Q \) of \( L \), \( \mathcal{O} \) includes an operation
\[ \tilde{Q} : P^+(\mathcal{V}) \rightarrow P^+(\mathcal{V}). \]
Example

Consider the two-valued Nmatrix $\mathcal{M}_1 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ for a language $L$ over $\{Q, \forall, \neg\}$, where $\mathcal{O}$ contains the following operations:

<table>
<thead>
<tr>
<th>H</th>
<th>$\tilde{Q}[H]$</th>
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<tbody>
<tr>
<td>${t}$</td>
<td>${t}$</td>
</tr>
<tr>
<td>${t,f}$</td>
<td>${t,f}$</td>
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<tr>
<td>${f}$</td>
<td>${f}$</td>
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</table>

<table>
<thead>
<tr>
<th>H</th>
<th>$\tilde{\forall}[H]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${t}$</td>
<td>${t}$</td>
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<tr>
<td>${t,f}$</td>
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<td>${f}$</td>
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</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\neg a$</th>
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</thead>
<tbody>
<tr>
<td>t</td>
<td>${t,f}$</td>
</tr>
<tr>
<td>f</td>
<td>${t}$</td>
</tr>
</tbody>
</table>
**$L$-structures**

An $L$-structure for an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a pair $S = \langle D, I \rangle$, where $D$ is a non-empty domain and $I$ satisfies the following properties:

- For every constant $c$ of $L$: $I[c] \in D$.
- For an $n$-ary predicate symbol $p$ of $L$: $I[p] : D^n \to \mathcal{V}$.
- For every $n$-ary function symbol $f$ of $L$: $I[f] : D^n \to D$.

$I$ is extended to interpret closed terms of $L$ as follows:

$$I[f(t_1, ..., t_n)] = I[f][I[t_1], ..., I[t_n]]$$

For $S = \langle D, I \rangle$, the language extended by individual constants $\{\bar{a} \mid a \in D\}$ is denoted by $L(D)$. $I$ is extended to interpret closed terms of $L(D)$: $I(\bar{a}) = a$. 
Naive Definition of Valuations in Nmatrices

Let \( S = \langle D, I \rangle \) be an \( L \)-structure. A valuation in an Nmatrix \( \mathcal{M} = \langle V, \mathcal{D}, \mathcal{O} \rangle \) for \( L \) is a function \( v \) from closed sentences of \( L(D) \) to \( V \), satisfying:

- \( v[p[t_1, \ldots, t_n]] = I[p][I[t_1], \ldots, I[t_n]]. \)
- \( v[\diamond[\psi_1, \ldots, \psi_n]] \in \tilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]. \)
- \( v[\mathcal{Q}x\psi] \in \tilde{\mathcal{Q}}[\{v[\psi\{\bar{a}/x\}] \mid a \in D\}]. \)
Reflection of the Problem of $\alpha$-Equivalence

Two $\alpha$-equivalent sentences are not necessarily assigned the same truth-value:

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<tr>
<th>H</th>
<th>$\forall[H]$</th>
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<td>{t}</td>
<td>{t}</td>
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<td>{t,f}</td>
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<tr>
<td>t</td>
<td>{t,f}</td>
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<tr>
<td>f</td>
<td>{t}</td>
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</tbody>
</table>

Let $S = \langle \{a, b\}, I \rangle$, $I[p][a] = I[p][b] = t$.

Consider: $\neg \forall x p(x)$ and $\neg \forall y p(y)$
The Definition of a Non-deterministic Valuation - Correction

Let $S = \langle D, I \rangle$ be an $L$-structure. A valuation in an Nmatrix $M = \langle V, D, O \rangle$ for $L$ is a function $v$ from closed sentences of $L(D)$ to $V$ satisfying:

- If $\psi_1 \equiv_\alpha \psi_2$, then $v[\psi_1] = v[\psi_2]$.

- $v[p[t_1, \ldots, t_n]] = I[p][I[t_1], \ldots, I[t_n]]$.

- $v[\diamond[\psi_1, \ldots, \psi_n]] \in \tilde{\diamond}[v[\psi_1], \ldots, v[\psi_n]]$.

- $v[Qx\psi] \in \tilde{Q}\{v[\psi\{a/x\}] | a \in D\}$.
Other Problems

- Terms denoting the same objects cannot be used interchangeably.

- Void quantification for first-order quantifiers $\forall$ and $\exists$ (e.g., first-order paraconsistent logics of da Costa).

  Consider: (i) $\neg p(c)$ and $\neg p(d)$, (ii) $\neg \forall x p(c)$ and $\neg p(c)$.

  Solution: add appropriate congruence relations.

  For instance, $A \sim_{\text{void}} Qx A$ if $x \notin Fv(A)$. 
Nmatrices with \( n \)-ary Quantifiers

- An \( n \)-ary quantifier \( Q \) in an Nmatrix \( \mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \) is interpreted by a function \( \tilde{Q} : P^+ (\mathcal{V}^n) \to P^+ (\mathcal{V}) \).

- Example: for every \( \mathcal{E} \in P^+ (\{t, f\}^2) \):

\[
\tilde{\forall} [\mathcal{E}] = \begin{cases} 
\{t\} & \text{if } \langle t, f \rangle \notin \mathcal{E} \\
\{f\} & \text{otherwise}
\end{cases}

\tilde{\exists} [\mathcal{E}] = \begin{cases} 
\{t\} & \text{if } \langle t, t \rangle \in \mathcal{E} \\
\{f\} & \text{otherwise}
\end{cases}

The definition of an \( \mathcal{M} \)-valuation \( v \) is now modified as follows:

\[
v[Qx(\psi_1, \ldots, \psi_n)] \in \tilde{Q}_\mathcal{M}[\{v[\psi_1 \{\bar{a}/x\}], \ldots, v(\psi_n \{\bar{a}/x\}) \mid a \in D\}]
\]
Example

<table>
<thead>
<tr>
<th>H</th>
<th>$\forall(H)$</th>
<th>$\exists(H)$</th>
<th>$\tilde{Q}_2(H)$</th>
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<tbody>
<tr>
<td>${\langle t, t \rangle}$</td>
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<td>${\langle t, f \rangle}$</td>
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<td>${t}$</td>
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Construction of a Characteristic 2Nmatrix

\[ \begin{array}{c}
\Rightarrow p(c_1) \quad p(c_2) \Rightarrow \\
\Rightarrow Qxp(x) \\
\Rightarrow p(y) \\
\Rightarrow Qxp(x) \Rightarrow
\end{array} \]

\[ \tilde{Q}[\{t\}] = ? \]

\[ \tilde{Q}[\{t, f\}] = ? \]

\[ \tilde{Q}[\{f\}] = ? \]

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<tr>
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<tr>
<td>{f, t}</td>
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<td>{f}</td>
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</table>
Construction of a Characteristic 2Nmatrix

\[
\Rightarrow p(c_1) \quad p(c_2) \Rightarrow \quad \Rightarrow p(y)
\]

\[\Rightarrow Qxp(x) \Rightarrow \quad Qxp(x) \Rightarrow\]

\[\tilde{Q}[\{t\}] = \{f\} \quad (\{\Rightarrow p(y)\} \text{ is valid in a structure with distribution } \{t\})\]

\[\tilde{Q}[\{t, f\}] = ?\]

\[\tilde{Q}[\{f\}] = ?\]

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Construction of a Characteristic 2Nmatrix

\[
\begin{align*}
  & \Rightarrow p(c_1) \quad p(c_2) \Rightarrow \\
  & \Rightarrow Qxp(x) \\
  & \Rightarrow p(y) \\
  & \Rightarrow Qxp(x) \Rightarrow
\end{align*}
\]

\(\tilde{Q}[\{t\}] = \{f\}\) (\(\{\Rightarrow p(y)\}\) is valid in a structure with distribution \(\{t\}\))

\(\tilde{Q}[\{t, f\}] = \{t\}\) (\(\{\Rightarrow p(c_1), p(c_2) \Rightarrow\}\) is valid in a structure with distribution \(\{f, t\}\))

\(\tilde{Q}[\{f\}] = ?\)

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<td>({f, t})</td>
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<td>({f})</td>
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</table>
How to Construct a Characteristic 2Nmatrix?

\[
\Rightarrow p(c_1)\quad p(c_2) \Rightarrow \quad \Rightarrow p(y) \quad \Rightarrow Qxp(x) \Rightarrow Qxp(x) \Rightarrow
\]

\(\tilde{Q}[\{t\}] = \{f\}\) (\(\Rightarrow p(y)\) is valid in a structure with distribution \(\{t\}\))

\(\tilde{Q}[\{t, f\}] = \{t\}\) (\(\Rightarrow p(c_1), p(c_2) \Rightarrow\) is valid in a structure with distribution \(\{f, t\}\))

\(\tilde{Q}[\{f\}] = \{f, t\}\) (There is no relevant rule)

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<tbody>
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<td>({f})</td>
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<tr>
<td>({f, t})</td>
<td>({t})</td>
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<tr>
<td>({f})</td>
<td>({t, f})</td>
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</table>
For every canonical calculus $G$:

- $G$ is coherent iff it has a (strongly) characteristic 2Nmatrix.
- If $G$ is coherent, then it admits cut-elimination.
- But cut-elimination does not imply coherence! *We can construct a calculus with binary quantifiers which is not coherent, but the only sequents provable in it are logical axioms.*
Exact Correspondence Still Holds for Strong Cut-elimination

The following statements concerning a canonical system $G$ with $n$-ary quantifiers are equivalent:

1. $G$ is coherent.
2. $G$ has a strongly characteristic 2Nmatrix.